SPHERICAL HARMONICS
&
PARTIAL DIFFERENTIAL EQUATIONS

FOR

POST-GRADUATE CLASSES

BY

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PREFACE

This little book embodies my lectures given in my classes for over twenty years.

I am grateful to my Colleague Professor S. C. Mittal for his useful suggestions and the kind interest that he has taken in the publication of the book.

I am also grateful to Shri Dinesh Krishna B. Sc. for the trouble that he has taken in correcting the proof sheets.

It shall give me great satisfaction if the book is found useful by students and teachers.

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Partial Differential Equations of Second and Higher Orders.

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Spherical Harmonics

FIRST LECTURE

Integration in Series, Legendre's Equation, $P_n(x)$, $Q_n(x)$.

To illustrate integration in series let us take the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)\,y=0\,\ldots\ldots\ldots(A)$$

This equation is known as Legendre's equation. It can be integrated in series of ascending or descending powers of x. The solution in descending powers of x is more important than the one in ascending powers. To find it out let us take

$$y=\sum_{r=0}^{\infty} a_r\, x^{k-r}$$

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then $$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r\,(k-r)\,x^{k-r-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r\,(k-r)(k-r-1)\,x^{k-r-2}$$

Substituting in (A) we get

$$\sum_{r=0}^{\infty}\left[(1-x^2)(k-r)(k-r-1)\,x^{k-r-2}-2(k-r)\,x^{k-r}+n(n+1)x^{k-r}\right]a_r=0$$

$$\sum_{r=0}^{\infty}\left[(k-r)(k-r-1)\,x^{k-r-2}+\{n(n+1)-(k-r)(k-r+1)\}\,x^{k-r}\right]a_r=0\,\ldots\ldots\ldots(B)$$

Now (B) being an identity we can equate to zero the coefficients of various powers of x.

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Equating to zero the coefficient of the highest power of x that is of x^k we have

$$a_0\{n(n+1)-k(k+1)\}=0.$$

Now $a_0 \neq 0$ as it is the coefficient of the term with which we begin to write the series, hence

$$\{n(n+1)-k(k+1)\}=0 \ldots\ldots\ldots (C)$$

$$\text{or}\quad (n-k)(n+k+1)=0 \ldots\ldots\ldots (C)$$

This gives the value of k either as

$$k=n, \text{ or as } k=-n-1 \ldots\ldots\ldots (D).$$

Equating to zero the coefficient of the next lower power of x that is of x^{k-1} in (B) we get

$$a_1\{n(n+1)-(k-1)k\}=0$$

Now $\{n(n+1)-(k-1)k\} \neq 0$ by virtue of (D) hence

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$$a_1=0 \ldots\ldots\ldots (E).$$

Equating to zero the coefficient of the general term namely that involving x^{k-r} in (B) we get

$$(k-r+2)(k-r+1)a_{r-2}+\{n(n+1)-(k-r)(k-r+1)\}a_r=0$$

$$\text{or}\quad a_r=-\frac{(k-r+2)(k-r+1)}{(n-k+r)(n+k-r+1)}\,a_{r-2} \quad \ldots\ \ldots\ \ldots\ \ldots\ (F).$$

Results (D) and (F) are important. The former determines the index and the latter gives the relation between the coefficients a's. Combining both these

$$a_r=-\frac{(n-r+2)(n-r+1)}{r\,.\,(2n-r+1)}\,a_{r-2} \quad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ (G)$$

when $k=n$ and

$$a_r=\frac{(n+r-1)(n+r)}{(2n+1+r)(r)}\,a_{r-2} \quad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ (H)$$

when $k=-n-1$

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Now, a_1 being zero from (E), we see from (F) that a_3, a_5,……… a_{2r+1},…………… are all zeroes, and a_2, a_4,………… a_{2r}, ……… can be found out in terms of a_0 which is an arbitrary constant.

Taking $k=n$ and the values of coefficients as determined from (G) and substituting in $y=\sum_{r=0}^{\infty} a_r x^{k-r}$ we get

$$y=a_0\left[x^n-\frac{n(n-1)}{2\,.\,(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}x^{n-4}+\ldots\ldots\right]\ldots\ldots\ldots(\alpha)$$

as one solution.

Taking $k=-n-1$ and the values of coefficients as determined from (H) we get

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$$y=a_0\left[x^{-n-}+\frac{(n+1)(n+2)}{2\,.\,(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2\,.\,4\,.\,(2n+3)(2n+5)}x^{-n-5}+\ldots\ldots\right]\ldots\ldots(\beta)$$

as the other solution of our differential equation where a_0 in both cases is an arbitrary constant.

With n a positive integer and $a_0=\dfrac{1\,.\,3\,.\,5\ldots(2n-1)}{\lfloor n}$,

the solution (α) is called $P_n(x)$.

$P_n(x)$ is a terminating series and gives what are called Legendre's Polynomials.

With n a positive integer and $a_0=\dfrac{\lfloor n}{1\,.\,3\,.\,5\ldots\ldots(2n+1)}$

the solution (β) is called $Q_n(x)$ and gives what are known as Legendre's functions of the second kind.

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We have thus

$$P_n(x)=\frac{1.3.5...(2n-1)}{\lfloor n}\Big[x^n-\frac{n(n-1)}{2.(2n-1)}x^{n-2}$$

$$+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}......$$

$$Q_n(x)=\frac{\lfloor n}{1.3.5...(2n+1)}\Big[x^{-n-1}+\frac{(n+1)(n+2)}{2.(2n+3)}x^{-n...}$$

$$+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5}+......+$$

The most general solution of the Legendre's equation is $y=AP_n(x)+BQ_n(x)$ where A and B are arb constants.

In the next few lectures we shall be more concerned finding out various properties, various series and v integrals for $P_n(x)$.

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Work for the next day.

1. With the help of lecture notes, write in fair han lecture of to-day.

2. Show that

$$P_0(\mu)=1,\ P_1(\mu)=\mu,\ P_2(\mu)=\frac{3\mu^2-1}{2}$$

$$P_3(\mu)=\frac{5\mu^3-3\mu}{2}$$

3. Show that

$$P_n(-\mu)=(-1)^nP_n(\mu).$$

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SECOND LECTURE

Generating function for $P_n(\mu)$; $P_n(1)=1$; Rodrigue's formula.

2. $P_n(\mu)$ is the coefficient of h^n in the expansion of $(1-2\mu h+h^2)^{-\frac{1}{2}}$

Now $(1-2\mu h+h^2)^{-\frac{1}{2}} = \{1-h(2\mu-h)\}^{-\frac{1}{2}}$

$$=1+\tfrac{1}{2}h(2\mu-h)+\frac{1\,.\,3}{2\,.\,4}h^2(2\mu-h)^2+\ldots\ldots$$

$$+\frac{1\,.\,3\,.\,5\ldots 2n-1}{2\,.\,4\,.\,6\ldots 2n}h^n(2\mu-h)^n+\ldots\ldots\ldots$$

for sufficiently small value of h

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The coefficient of h^n in this expansion is

$$\frac{1\,.\,3\,.\,5\,.\,.\,2n-1}{2\,.\,4\,.\,6\,.\,\,2n}(2\mu)^n-\frac{1\,.\,3\,.\,5\,.\,.\,2n-3}{2\,.\,4\ldots 2n-2}.(2\mu)^{n-2}\,.\,{}^{n-1}C_1$$

$$+\frac{1\,.\,3\,.\,5\ldots 2n-5}{2\,.\,4\,.\,6\,.\,.\,2n-4}(2\mu)^{n-4}\,.\,{}^{n-2}C_2$$

$$-\frac{1\,.\,3\,.\,5\ldots.2n-7}{2\,.\,4\,.\,6\ldots 2n-6}(2\mu)^{n-6}\;{}^{n-3}C_3+\ldots\ldots\ldots$$

Or $\dfrac{1\,.\,3\,.\,5\ldots 2n-1}{\lfloor n}\Big\{\mu^n-\dfrac{n(n-1)}{2\,.\,(2n-1)}\mu^{n-2}$

$$+\frac{n(n-1)(n-2)(n-3)\mu^{n-4}}{2\,.\,2\,.\,\lfloor 2\;(2n-1)(2n-3)}$$

$$-\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2^3\lfloor 3\;(2n-1)(2n-3)(2n-5)}\mu^{n-6}+\ldots\ldots\ldots\Big\}$$

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Or $\dfrac{1\,.\,3\,.\,5\ldots(2n-1)}{\lfloor \underline{n}}\Big\{\mu^n-\dfrac{n(n-1)}{2\,.\,(2n-1)}\mu^{n-2}$

$+\dfrac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}\mu^{n-4}$

$-\dfrac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\,.\,4\,.\,6\,(2n-1)(2n-3)(2n-5)}\mu^{n-6}+\ldots\ldots\ldots\Big\}$

This is $P_n(\mu)$

Thus $\sum_{n=0}^{\infty} h^n P_n(\mu)=(1-2\mu h+h^2)^{-\frac{1}{2}}$

3. From the last article $P_n(1)$ is the coefficient of h^n in the expansion of $(1-2h+h^2)^{-\frac{1}{2}}$

or in the expansion of $(1-h)^{-1}$

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or in $1+h+h^2+\ldots\ldots+h^n+\ldots\ldots\ldots\ldots$

This coefficient is 1

hence $P_n(1)=1$.

This is a distinguishing property of Legendre's polynomia

4. **Rodrigue's formula.**

Let us form a differential equation in y it being give that $y=(\mu^2-1)^n$

Now $\dfrac{dy}{d\mu}=n(\mu^2-1)^{n-1}(2\mu)$

or $(\mu^2-1)\dfrac{dy}{d\mu}=2n\mu y\ldots\ldots\ldots\ldots\ldots(1)$

Differentiating (1) $(n+1)$ times by Lebnitz's theorem

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$$(\mu^2-1)\ \frac{d^{n+2}y}{d\mu^{n+2}}+{}^{n+1}C_1(2\mu)\ \frac{d^{n+1}y}{d\mu^{n+1}}+{}^{n+1}C_2(2)\frac{d^n y}{d\mu^n}$$

$$=2n\left[\mu\frac{d^{n+1}y}{d\mu^{n+1}}+{}^{n+1}C_1\frac{d^n y}{d\mu^n}\right]$$

or $$(\mu^2-1)\frac{d^{n+2}y}{d\mu^{n+2}}+2\mu\ \frac{d^{n+1}y}{d\mu^{n+1}}-n(n+1)\ \frac{d^n y}{d\mu^n}=0$$

or $$(\mu^2-1)\ \frac{d^2Z}{d\mu^2}+2\mu\ \frac{dZ}{d\mu}-n(n+1)Z=0. \qquad (2)$$

where $Z=\frac{d^n y}{d\mu^n}$

Now (2) is Legendre's equation

It is satisfied by CZ where C is a constant

Thus CZ or $C\ \frac{d^n y}{d\mu^n}$ is a solution of Legendre's equation

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and this will be $P_n(\mu)$ if $C\frac{d^n y}{d\mu^n}=1$ when $\mu=1$.

or if $C\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n=1$ when $\mu=1$.

Considering $(\mu^2-1)^n$ as $(\mu-1)^n(\mu+1)^n$ and differentiating n times by Lebnitz's theorem and then putting $\mu=1$ we get

$C.2^n \lfloor n =1$, so that $C=\frac{1}{2^n\lfloor n}$

With this value of C, CZ is $P_n(\mu)$

or $$\frac{1}{2^n\lfloor n}\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n=P_n(\mu).$$

This is Rodrigue's Formula.

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2. Work for the next day.

1. Integrate the hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{dy}{dx}-\alpha\beta y=0$$

in series of ascending powers of x and show that the comple primitive is.

$$A\ F(\alpha,\beta,\gamma,x)+Bx^{1-\gamma}F(\alpha+1-\gamma,\ \beta+1-\gamma,\ 2-\gamma,$$

where A and B are arbitrary constants and $F(\alpha, \beta, \gamma,$ stands for the series.

$$1+\frac{\alpha\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\,\beta\,(\beta+1)}{1\,.\,2\,\gamma\,(\gamma+1)}x^2$$

$$+\frac{\alpha(\alpha+1)(\alpha+2)\,\beta\,(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,\gamma\,(\gamma+1(\gamma+2)}x^3+\dots\dots+\dots\dots$$

2. Show that $P_{2m+1}(0)=0$

$$\text{and}\quad P_{2m}(0)=(-1)^m\frac{1\,.\,3\,.\,5\dots\dots2m-1}{2\,.\,4\,.\,6\dots\dots2m}$$

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THIRD LECTURE

[Various trigonometrical series for $P_n(\mu)$]

5. $P_n(\cos\theta)$ as a series in cosines of multiples of θ.

We know from Art 2

$$\sum_{n=0}^{\infty} h^n P_n(\cos\theta) = (1-2\cos\theta\, h+h^2)^{-\frac{1}{2}}$$

$$=\left\{1-\left(e^{i\theta}+e^{-i\theta}\right)h+e^{i\theta}\cdot e^{-i\theta}\cdot h^2\right\}^{-\frac{1}{2}}$$

$$=\left(1-e^{i\theta}h\right)^{-\frac{1}{2}}\left(1-e^{-i\theta}h\right)^{-\frac{1}{2}} \quad \ldots \quad \ldots \quad (1)$$

Now $\left(1-e^{i\theta}h\right)^{-\frac{1}{2}}=1+\frac{1}{2}e^{i\theta}h+\frac{1\cdot 3}{2\cdot 4}e^{2i\theta}h^2$

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$$+\ldots\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}e^{ni\theta}h^n+\ldots\ldots\ldots$$

$$\left(1-e^{-i\theta}h\right)^{-\frac{1}{2}}=1+\frac{1}{2}e^{-i\theta}h+\frac{1\cdot 3}{2\cdot 4}e^{-2i\theta}h^2$$

$$+\ldots\ldots+\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}e^{-ni\theta}h^n+\ldots\ldots\ldots$$

The coefficient of h^n in the product of the above two expansions is,

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\Big[\left(e^{in\theta}+e^{-in\theta}\right)$$

$$+\frac{2n}{2n-1}\left(e^{i(n-2)\theta}+e^{-i(n-2)\theta}\right)\cdot\frac{1}{2}$$

$$+\frac{2n(2n-2)}{(2n-1)(2n-3)}\left(e^{i(n-4)\theta}+e^{-i(n-4)\theta}\right)\frac{1\cdot 3}{2\cdot 4}+\ldots\ldots\Big]$$

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or $\frac{1 \cdot 3 \cdot 5 \ldots 2n-1}{2 \cdot 4 \cdot 6 \ldots 2n}\Big[2 \cos (n\theta) + \frac{n}{2n-1} \cdot 2 \cos (n-2$

$$+\frac{n(n-1) \cdot 1 \cdot 3}{(2n-1)(2n-3) \cdot 1 \cdot 2} 2 \cdot \cos (n-4)\theta$$

$$+\frac{n(n-1)(n-2)\ 1 \cdot 3 \cdot 5}{(2n-2)(2n-3)(2n-5) \cdot 1 \cdot 2 \cdot 3} 2 \cdot \cos(n-6)\theta + \ldots$$

By equating coefficients of h^n on both sides of (1) we have

$$P_n(\cos\theta) = \frac{1 \cdot 3 \cdot 5 \ldots 2n-1}{2 \cdot 4 \cdot 6 \ldots 2n}\Big[2 \cos n\theta + 2 \ldots \frac{1 \cdot n}{1 \cdot (2n-1)}$$

$$\cos (n-2$$

$$+ 2 \cdot \frac{1 \cdot 3 \cdot n(n-1)}{1 \cdot 2\,(2n-1)(2n-3)} \cos (n-4)\theta$$

$$+2 \cdot \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)} \cos (n-6)$$

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$$+ \ldots\ldots\ldots\ldots\Big]$$

6. $\left(1-2\cos\theta\, h+h^2\right)^{-\frac{1}{2}} = \left\{(1-h)^2+4h \sin^2\frac{\theta}{2}\right\}^{-}$

$$=\frac{1}{1-h}\left\{1+\frac{4h \sin^2\frac{\theta}{2}}{(1-h)^2}\right\}^{-\frac{1}{2}}$$

$$=\frac{1}{1-h}+\sum_{r=1}^{\infty}(-1)^r\frac{1 \cdot 3 \cdot 5 \ldots 2r-1}{2 \cdot 4 \cdot 6 \ldots 2r} \cdot \frac{2^r \cdot 2^r \cdot \sin^{2r}\frac{\theta}{2}}{(1-h)^{2r+1}}$$

Equating coefficient of h^n on both sides,

$$P_n(\cos\theta)=1+\sum_{r=1}^{\infty}(-1)^r \cdot \frac{1 \cdot 3 \cdot 5 \ldots 2r-1}{\lfloor r} \cdot 2^r \sin^{2r}\frac{\theta}{2}$$

$$\frac{(2r+1)(2r+2)\ldots\ldots(2r+1+n-r-1)}{\lfloor n-r}$$

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$$1+\sum_{r=1}^{\infty}(-1)^r\cdot\frac{1.3.5\ldots 2r-1}{\lfloor r}\cdot\frac{\lfloor n+r}{\lfloor 2r\ \lfloor n-r}\cdot\times$$

$$2^r\sin^{2r}\frac{\theta}{2}.$$

$$=1+\sum_{r=1}^{\infty}(-1)^r\cdot\frac{1}{\lfloor r\lfloor r}\ (n-r+1)(n-r+2)\ldots\ldots(n+r)\times$$

$$\sin^{2r}\frac{\theta}{2}$$

$$=1-\frac{n(n+1)}{\lfloor 1\ \lfloor 1}\sin^2\frac{\theta}{2}+\frac{(n-1)(n)(n+1)(n+2)}{\lfloor 2\ \lfloor 2}\sin^4\frac{\theta}{2}$$

$$-\frac{(n-2)(n-1)(n)(n+1)(n+2)(n+3)}{\lfloor 3\ \lfloor 3}\sin^6\frac{\theta}{2}+\ldots\ldots\ldots$$

$$=F\left(n+1,\ -n,\ 1,\ \sin^2\frac{\theta}{2}\right).$$

Putting $\theta+\pi$ for θ and remembering that $P_n(-\mu)=(-1)^n$ $P_n\ (\mu)$, we see that

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$$P_n\ (\cos\theta)=(-1)^n\ F\left(n+1,\ -n,\ 1,\ \cos^2\frac{\theta}{2}\right).$$

Work for the next day. III

1. Show that the Legendre's equation

$$(1-\mu^2)\frac{d^2y}{d\mu^2}-2\mu\frac{dy}{d\mu}+n(n+1)y=0$$

changes into the hypergeometric form

$$x\ (1-x)\ \frac{d^2y}{dx^2}+(\tfrac{1}{2}-\tfrac{3}{2}x)\frac{dy}{dx}+\frac{n\ (n+1)}{4}y.$$

by the transformation $\mu^2=x$

Hence show by comparison that its complete primitive is

$$A\ F\left(-\frac{n}{2},\ \frac{n+1}{2},\ \tfrac{1}{2},\ \mu^2\right)+B\mu F\left(-\frac{n-1}{2},\ \frac{n+2}{2},\ \tfrac{3}{2},\ \mu^2\right)$$

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FOURTH LECTURE

[Zeroes of $P_n(\mu)$]

7. All the zeroes of $P_n(\mu)$ are real and lie between $+1$ and -1.

Taking μ to be real let

$$F(\mu)=(\mu-1)^n(\mu+1)^n$$

Applying Libnitz's theorem we see that all the differenti coefficients of $F(\mu)$ upto the $(n-1)^{th}$ vanish both for $\mu=-$ and for $\mu=+1$, but the n^{th} defferential coefficient vanish for neither.

Applying Rolle's theorem in succession we see that

$$\left(\frac{d}{d\mu}\right)^n\left\{(\mu-1)^n(\mu+1)^n\right\}$$

vanishes for at least n values of μ lying between -1 and $+1$
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And as it is of the n^{th} degree in μ it can not vanish fo any value outside these n values.

Hence all the zeroes of $P_n(\mu)$ or of

$$\frac{1}{2^n \lfloor n}\left(\frac{d}{d\mu}\right)^n\left\{(\mu^2-1)^n\right\} \text{ lie}$$

between $\mu=-1$ and $\mu=1$ and are all real.

8. All the roots of $P_n(\mu)=0$ are different

If they are not all different at least two of them must b equal. Let their common value be α.

Then $P_n(\alpha)=0$ $P_n'(\alpha)=0$

From the Legendre's equation

$$(1-\mu^2)\frac{d^2P_n}{d\mu^2}-2\mu\frac{dP_n}{d\mu}+n(n+1)P_n=0$$

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$P_n''(\alpha)=0.$

and from the Legendre's equation differentialed once

$P_n'''(\alpha)=0.$

and then from the Legendre's equation differentialed twice

$P_n''''(\alpha)=0.$

and so on. Thus $P_n^{(n)}(\alpha)=0$

which is absurd as $P_n^{(n)}(\alpha)$ is always a constant. Thus all the roots of $P_n(\mu)=0$ are different. Let them be $\alpha_1, \alpha_2, \alpha_3, \ldots$ ... ... Then since α is a root of $P_n(\mu)=0$, then $-\alpha$ is also a root of $P^n(\mu)=0$ because of the relation $P_n(-\alpha)=(-1)^n P_n(\alpha)$.

Thus $P_n(\mu)=A(\mu^2-\alpha_2^2)(\mu^2-\alpha_2^2)\ldots\ldots\ldots(\mu^2-\alpha_n^2)$ if n is even.

$$P_n(\mu)=B\mu(\mu^2-\beta_1^2)(\mu^2-\beta_2^2)\ldots\ldots(\mu^2-\beta^2_{\frac{n-1}{2}})$$ if n is odd.

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A and B being certain constants.

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Work for the next day IV

1. Show that the complete primitive of the hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2}+\left\{\gamma-(\alpha+\beta+1)x\right\}\frac{dy}{dx}-\alpha\beta y=0$$

is $AX^{-\alpha}F\left[\alpha, 1-\gamma+\alpha, 1-\beta+\alpha, \frac{1}{x}\right]$

$$+BX^{-\beta}F\left[\beta, 1-\gamma+\beta, 1-\alpha+\beta, \frac{1}{x}\right]$$

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From the above deduce that the complete primitive of the Legendre's equation is

$$A\mu^n F\left(-\frac{n}{2}, \frac{1-n}{2}, \tfrac{1}{2}-n, \frac{1}{\mu^2}\right)$$

$$+B\mu^{-n-1} F\left(\frac{n+1}{2}, \frac{n+2}{2}, \tfrac{3}{2}+n, \frac{1}{\mu^2}\right).$$

The first part will be P^n if n is a positive integer for suitable value of the constant A.

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FIFTH LECTURE

Laplace's definite integrals for $P_n(\mu)$.

9 We know

$$\int_0^\pi \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}} \text{ where } a^2 > b^2$$

Put $a=1-\mu h$, $b=h\sqrt{\mu^2-1}$

and therefore $a^2-b^2=1-2\mu h+h^2$

We get

$$\frac{\pi}{(1-2\mu h+h^2)^{\frac{1}{2}}} = \int_0^\pi \frac{d\phi}{1-h\mu \pm h\sqrt{\mu^2-1}\cos\phi}$$

$$= \int_0^\pi \left\{ \sum_{n=0}^{\infty} h^n(\mu \mp \sqrt{\mu^2-1}\cos\phi)^n \right\} d\phi$$

for sufficiently small values of h.

$$= \sum_{n=0}^{\infty} \int_0^\pi h^n(\mu \mp \sqrt{\mu^2-1}\cos\phi)^n \, d\phi$$

Equating coefficients of h^n we get

$$P_n(\mu) = \frac{1}{\pi}\int_0^\pi (\mu \mp \sqrt{\mu^2-1}\cos\phi)^n \, d\phi$$

This is the first of the Laplace's integrals.

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10 $$\int_0^\pi \frac{d\phi}{a-b\cos\phi}=\frac{\pi}{\sqrt{a^2-b^2}} \text{ where } a^2>b^2$$

put $a=\mu h-1$, $b=\mp h\sqrt{\mu^2-1}$

then $$\frac{\pi}{\sqrt{a^2-b^2}}=\frac{\pi}{\sqrt{1-2\mu h+h^2}}=\frac{\pi}{h\sqrt{1-\frac{2\mu}{h}+\frac{1}{h^2}}}$$

$$=\pi\sum_{n=0}^{\infty}\frac{1}{h^{n+1}}P_n(\mu)$$

$$=\int_0^\pi \frac{d\phi}{h(\mu\pm\sqrt{\mu^2-1}\cos\phi)-1}$$

$$=\int_0^\pi \left\{\frac{\left[1-\frac{1}{h(\mu\pm\sqrt{\mu^2-1}\cos\phi)}\right]^{-1}}{h(\mu\pm\sqrt{\mu^2-1}\cos\phi)}\right\}d\phi$$

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$$=\int_0^\pi \left\{\sum_0^\infty \frac{1}{h^{n+1}(\mu\pm\sqrt{\mu^2-1}\cos\phi)^{n+1}}\right\}d\phi$$

$$=\sum_{n=0}^{\infty}\int_0^\pi \frac{d\phi}{h^{n+1}(\mu\pm\sqrt{\mu^2-1}\cos\phi)^{n+1}}$$

picking coefficients of $\frac{1}{h^{n+1}}$ on both sides we have

$$P_n(\mu)=\frac{1}{\pi}\int_0^\pi \frac{d\phi}{(\mu\pm\sqrt{\mu^2-1}\cos\phi)^{n+1}}$$

This is the second of Laplace's integrals.
From these two integrals we easily see that

$$P_n(\mu)=P_{-n-1}(\mu).$$

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Work for the next day V.

1. Show that

$$P_n(\cos\theta)=\cos^{2n}\frac{\theta}{2}\ F\left(-n,\ -n,\ 1,\ -\tan^2\frac{\theta}{2}\right)$$

(Murphy)

2. The following is due to Eular

$$P_n(\cos\theta)=\cos^n\theta\ F\left(-\frac{n}{2}\ ,\ \frac{1}{2}-\frac{n}{2}\ ,\ 1,\ -\tan^2\theta\right)$$

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SIXTH LECTURE

[Recurrence formulae]

11. Let $V=(1-2\mu h+h^2)^{-\frac{1}{2}}$

then $V^2\,(1-2\mu h+h^2)=1$

Differentiating with respect to h and dividing by $2V$ we have

$$V\,(h-\mu)+\frac{dV}{dh}\,(1-2\mu h+h^2)=0$$

or $(h-\mu)\,\Sigma\, h^n\;P_n\;(\mu)+(1-2\mu h+h^2).\;\Sigma\;nh^{n-1}P_n(\mu)=0$

Equating coefficients of h^{n-1} to zero we get

$$P_{n-2}-\mu\,P_{n-1}+nP_n-2\mu(n-1)\,P_{n-1}+(n-2)\,P_{n-2}=0$$

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or $nP_n=(2n-1)\,\mu P_{n-1}-(n-1)\,P_{n-2}$ ....(1)

12. To prove

Now $(\mu^2-1)\dfrac{dP_n}{d\mu}=n(\mu P_n-P_{n-1})=-(n+1)(\mu P_n-P_{n+1})$

$$\mu P_n-P_{n-1}=\frac{1}{\pi}\mu\int_0^{\pi}(\mu+\sqrt{\mu^2-1}\,\cos\phi)^n\,d\phi$$

$$-\frac{1}{\pi}\int_0^{\pi}(\mu+\sqrt{\mu^2-1}\,\cos\phi)^{n-1}\,d\phi$$

$$=\frac{1}{\pi}\int_0^{\pi}(\mu+\sqrt{\mu^2-1}\,\cos\phi)^{n-1}\{\mu^2+\mu\sqrt{\mu^2-1}\,\cos\phi-1\}d\phi$$

$$=\frac{\mu^2-1}{\pi}\int_0^{\pi}(\mu+\sqrt{\mu^2-1}\,\cos\phi)^{n-1}\left(1+\frac{\mu\cos\phi}{\sqrt{\mu^2-1}}\right)d\phi$$

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$$=\frac{(\mu^2-1)}{n\pi}\ \frac{d}{d\mu}\left[\int_0^{\pi}(\mu+\sqrt{\mu^2-1}\ \cos\phi)^n\,d\phi\ \right]$$

$$=\frac{\mu^2-1}{n}\ \frac{dP_n^2}{d\mu}$$

Hence

$$\left.\begin{aligned}(\mu^2-1)\frac{dP_n}{d\mu}&=n(\mu P_n-P_{n-1})\ \ldots\ldots\ldots\ldots(A)\\(\mu^2-1)\frac{dP_n}{d\mu}&=-(n+1)(\mu P_n-P_{n+1})\ldots\ldots(B)\end{aligned}\right\}\ldots\ldots\ldots(II)$$

Since $P_n=P_{-n-1}$

13. We have

$$\frac{d}{d\mu}\left\{(\mu^2-1)\ \frac{dP_n}{d\mu}\right\}=n(n+1)P_n$$

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or $\frac{d}{d\mu}\left[n(\mu P_n-P_{n-1})\right]=n(n+1)P_n$ by..........(II)

or $\mu\frac{dP_n}{d\mu}-\frac{dP_{n-1}}{d\mu}=nP_n\ldots\ldots\ldots\ldots(III)$

also $-\mu\frac{dP_n}{d\mu}+\frac{dP_{n+1}}{d\mu}=(n+1)P_n\ldots\ldots\ldots\ldots(IV)$

since $P_n=P_{-n-1}$

By adding (iii) and (iv) we get

$$(2n+1)\ P_n=\frac{dP_{n+1}}{d\mu}-\frac{dP_{n-1}}{d\mu}\ \ldots\ldots\ldots\ldots\ldots(V)$$

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14. From (v) $\frac{dP_n}{d\mu}=(2n-1)P_{n-1}+\frac{dP_{n-2}}{d\mu}$

$$=(2n-1)P_{n-1}+(2n-5)P_{n-3}+\frac{dP_{n-4}}{d\mu}$$

by substituting for $\frac{dP_{n-2}}{d\mu}$ from (v)

Thus by repeated application of (v) we get

$$\frac{dP_n}{d\mu}=(2n-1)P_{n-1}+(2n-5)P_{n-3}+(2n-9)P_{n-5}+\ldots\ldots \text{(VI)}$$

the last term being $3P_1$ or P_0 according as n is even or odd.

This is called **Christoffel's** expansion.

From (II) (A and B) we have

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$$(n+1)(\mu^2-1)\frac{dP_n}{d\mu}=n(n+1)(\mu P_n-P_{n-1})$$

$$n(\mu^2-1)\frac{dP_n}{d\mu}=-(n+1)n\,.\,(\mu P_n-P_{n+1})$$

adding we get

$$(2n+1)(\mu^2-1)\frac{dP_n}{d\mu}=n(n+1)(P_{n+1}-P_{n-1})$$

or $(\mu^2-1)\frac{dP_n}{d\mu}=\frac{n(n+1)}{2n+1}(P_{n+1}-P_{n-1})$..................(VII)

This is called **Beltramis** result.

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Work for the next day VI.

1. If (u, ϕ, z) and (r, θ, ϕ) be the cylindrical and polar coordinates of the same point and if $\mu = \cos\theta$,

show that $P_n(\mu) = (-1)^n \dfrac{r^{n+1}}{\lfloor n} \left(\dfrac{\partial}{\partial z}\right)^n \left(\dfrac{1}{r}\right)$

(London B. Sc. 1926).

2. $\displaystyle\int_{-1}^{1} (1-x^2)\left(\frac{dP_n}{dx}\right)^2 dx = \frac{2n(n+1)}{2n+1}$ (Agra 1945)

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SEVENTH LECTURE

[Christoffel's Summation formula for the sum of the first n+1 terms of the series

$$\sum_{r=0}^{\infty}(2r+1)\, P_r(x)\, P_r(y),$$ **Orthogonal properties of** $P(\mu)$**]**

15. Now

$$(2r+1)x\, P_r(x)\, P_r(y) = P_r(y)\,[(r+1)\, P_{r+1}(x) + rP_{r-1}(x)]$$

$$(2r+1)y\, P_r(y)\, P_r(x) = P_r(x)\,[(r+1)\, P_{r+1}(y) + rP_{r-1}(y)]$$

Subtracting

$$\sum_{r=0}^{n}(2r+1)\, P_r(x)\, P_r(y).\,(x-y)$$

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$$=\sum_{r=0}^{n}(r+1)\{P_{r+1}(x)\, P_r(y) - P_{r+1}(y)\, P_r(x)$$

$$-r\{P_r(x)\, P_{r-1}(y) - P_r(y)\, P_{r-1}(x)\}\}$$

$$=(n+1)\{P_{n+1}(x)\, P_n(y) - P_{n+1}(y)\, P_n(x)\}$$

Hence $\sum_{r=0}^{n}(2r+1)\, P_r(x)\, P_r(y)$

$$=(n+1)\frac{P_{n+1}(x)\, P_n(y) - P_{n+1}(y)\, P_n(x)}{x-y}$$

16 $\int_{-1}^{1} P_m(\mu)\, P_n(\mu)\, d\mu = 0$ or $\frac{2}{2n+1}$, according as m and n are different or $m=n$

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$$\int_{-1}^{1} P_n(\mu)\, P_m(\mu) d\mu = \frac{1}{2^{m+n} \lfloor m \lfloor n} \times$$

$$\int_{-1}^{1} \left(\frac{d}{d\mu}\right)^n (\mu^2-1)^n \left(\frac{d}{d\mu}\right)^m (\mu^2-1)^m \, d\mu$$

$$= K\left[\left(\frac{d}{d\mu}\right)^m (\mu^2-1)^m \left(\frac{d}{d\mu}\right)^{n-1} (\mu^2-1)^n\right]_{-1}^{1}$$

$$- K\int_{-1}^{1} \left(\frac{d}{d\mu}\right)^{m+1} (\mu^2-1)^m \left(\frac{d}{d\mu}\right)^{n-1} (\mu^2-1)^n \, d\mu$$

Where $K = \frac{1}{2^{m+n}} \cdot \frac{1}{\lfloor n} \cdot \frac{1}{\lfloor m}$ www.dbraulibrary.org.in

The first portion vanishes at both the limits. We integrate the second portion by parts again and continue the process till we find the integral to be

$$(-1)^m \cdot K \int_{-1}^{1} \frac{d^{n-m}(\mu^2-1)^n}{d\mu^{n-m}} \cdot \frac{d^{2m}(\mu^2-1)^m}{d\mu^{2m}} d\mu$$

$$= (-1)^m \, K \, \lfloor 2m \int_{-1}^{1} \frac{d^{n-m}(\mu^2-1)^n}{d\mu^{n-m}} d\mu \quad \ldots\ldots\ldots(1)$$

$$= (-1)^m \cdot K \cdot \lfloor 2m \left[\frac{d^{n-m-1}(\mu^2-1)^n}{d\mu^{n-m}}\right]_{-1}^{1} = 0$$

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If $m=n$ the integral becomes from (1)

$$\int_{-1}^{1} P_n^2(\mu)\, d\mu = (-1)^n \cdot \frac{\lfloor 2n}{2^n \cdot 2^n \lfloor n \lfloor n} \int_{-1}^{1} (\mu^2-1)^n d\mu$$

$$= \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \lfloor n} \int_{-1}^{1} (1-\mu^2)^n \, d\mu$$

$$= \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \lfloor n} \int_{0}^{\pi} \sin^{2n+1}\theta \, d\theta$$

(where $\mu = \cos\theta$)

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$$= 2 \frac{1 \cdot 3 \cdot 5 \ldots\ldots 2n-1}{2^n \lfloor n} \int_{0}^{\pi/2} \sin^{2n+1}\theta \, d\theta.$$

$$= 2 \frac{1 \cdot 3 \cdot 5 \ldots\ldots 2n-1}{2^n \lfloor n} \cdot \frac{2n\,(2n-2) \quad 2}{(2n+1)(2n-1)\ldots 3}$$

$$= \frac{2}{2n+1}$$

Examples on Legendre's Polynomials

1. Prove that

$$\frac{1-z^2}{(1-2xz+z^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} (2n+1)\, P_n(x)\, z^n$$

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2. Prove

$$\frac{1+z}{z(1-2xz+z^2)^{\frac{1}{2}}}-\frac{1}{z}=\sum_{n=0}^{\infty}\{P_n(x)+P_{n+1}(x)\}z^n$$

3 By inserting $\cot\theta'=\cot\theta-h\operatorname{cosec}\theta$ and expanding θ' in powers of h by Taylor's theorem show that

$$P_n(\cos\theta)=\frac{(-1)^n}{\lfloor n}\operatorname{cosec}^{n+1}\theta\,\frac{d^n(\sin\theta)}{d(\cot\theta)^n}$$

4. Show that

$$\int_{-1}^{1}\left(\frac{dP_n}{d\mu}\right)^2 d\mu=n(n+1)$$

5. Show that

$$1+\frac{P_1}{2}+\frac{P_2}{3}+\frac{P_3}{4}+\ldots\ldots=\log\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

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6. Prove that

$$\begin{vmatrix} 1 & P_1\ldots\ldots\ldots\ldots P_n \\ P_1 & P_2\ldots\ldots\ldots\ldots P_{n+1} \\ \ldots & \ldots\ldots\ldots\ldots \\ \ldots & \ldots\ldots\ldots\ldots \\ P^n & P_{n+1}\ldots\ldots\ldots P_{2n} \end{vmatrix}$$

is a numerical multiple of $(x^2-1)^{\frac{n(n+1)}{2}}$

7. Show that $P_0^2+3P_1^2+5P_2^2+\ldots\ldots(2n+1)P_n^2$

$$=(n+1)\{P_nP'_{n+1}-P_{n+1}P'_n\}$$

[Taking $y=x+\varepsilon$ in christoffel's formula and expanding Taylor's theorm and taking limits as $\varepsilon\to 0$ we get the result].

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8. Show that $(n+1)^2 P^2_n - (\mu^2-1) P'^2_n$

$$= (n+1)(P_n P'_{n+1} - P_{n+1} P'_n)$$

From (II) B, multiplied by P'_n and from (IV) multiplied by $(n+1) P_n$ and subtracting we get the result.

9. Show that

$$P_0^2 + 3P_1^2 + 5P_2^2 + \dots\dots\dots + (2n+1)P_n^2$$

$$= (n+1)^2 P^2_n - (\mu^2-1) P'^2_n.$$

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EIGHTH LECTURE

[Integration of Bessel's equation in series, Bessel's functions.]

17. Let us integrate the Bessel's differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0$$

Put $y=\sum_{r=0}^{\infty} a_r x^{k+r}$

then $\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1}$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}$$

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Substituting we have

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2}+\sum_{r=0}^{\infty} a_r (k+r) x^{k+r-2}$$

$$+\sum_{r=0}^{\infty} a_r x^{k+r}-\sum_{r=0}^{\infty} a_r n^2 x^{k+r-2}=0.$$

or $\sum_{r=0}^{\infty} a_r \{(k+r)^2-n^2\} x^{k+r-2}+\sum_{r=0}^{\infty} a_r x^{k+r}=0$

Equating to zero the coefficient of x^{k-2} which is the lowest power of x we have

$a_0 (k^2-n^2)=0$ which gives $k=\pm n$.........(1)

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Equating to zero the coefficient of x^{k-1} which is the next higher power of x we have

$$a_1 \{(1+k)^2-n^2\}=0$$

As $(1+k)^2-n^2 \neq 0$ by virtue of (1)

So $a_1=0$ ....(2)

Equating to zero the coefficient of x^{k+r} we get

$$a_{r+2}\{(k+r+2)^2-n^2\}+a_r=0 \quad \ldots\ldots\ldots\ldots (3)$$

or $\quad a_{r+2}=-\dfrac{a_r}{(n+r+2)^2-n^2}$ for $k=n$ .... (4)

$\quad a_{r+2}=-\dfrac{a_r}{(-n+r+2)^2-n^2}$ for $k=-n$ .... (5)

From (4) $a_{r+2}=-\dfrac{a_r}{(r+2)(2n+r+2)}$ ....(6)

From (2) and (6) $a_1=a_3=a_5=\ldots\ldots\ldots=0$

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Giving to r values 0, 2, 4, 6, ...we get corresponding a's which on substitution give us our solution as the series

$$y=a_0\, x^n \left[1-\frac{x^2}{2\,.\,2\,(n+1)}+\frac{x^4}{2\,.\,4\,.\,2^2\,(n+1)(n+2)} - \ldots\ldots+(-1)^r \frac{x^{2r}}{2^r\, \lfloor\underline{r}\,.\,2^r\,.\,(n+1)(n+2)\ldots(n+r)} + \ldots\ldots\ldots \right] \quad (7)$$

where a_0 is an arbitrary constant.

The other solution is obtained by replacing n by $-n$ in (7).

The solution as given by (7) is called $J_n(x)$ when

$$a_0=\frac{1}{2^n \Gamma\,(n+1)}\,.$$

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us

$$(x) = \frac{x^n}{2^n \Gamma(n+1)} \times$$

$$\left[\sum_{r=0}^{\infty} (-1)^r \cdot \left(\frac{x}{2}\right)^{2r} \frac{1}{\lfloor\!\underline{r} \cdot (n+1)(n+2)\dots(n+r)} \right]$$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\lfloor\!\underline{r}\; \Gamma(n+r+1)}.$$

d $\quad J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\lfloor\!\underline{r}\; \Gamma(-n+r+1)}$

e complete primitive of Bessel's equation is

$$A\, J_n(x) + B\, J_{-n}(x).$$

To prove that $J_{-n}(x) = (-1)^n J_n(x)$, n being integral ositive we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{\lfloor\!\underline{r}\; \Gamma(-n+r+1)}$$

re in the denominator of all terms which precede the e have the argument of Γ a negative integer or zero all the terms which precede the nth vanish as the inator becomes infinite.

nce

$$(x) = \sum_{r=n}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{\lfloor\!\underline{r}\; \Gamma(-n+r+1)}$$

$$\sum_{s=0}^{\infty} (-1)^{s+n} \left(\frac{x}{2}\right)^{-n+2(s+n)} \cdot \frac{1}{\lfloor\!\underline{n+s}\; \Gamma(-n+s+n+1)}$$

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$$= \sum_{s=0}^{\infty} (-1)^{s+n}\left(\frac{x}{2}\right)^{2s+n} \cdot \frac{1}{\lfloor n+s \; \Gamma(s+1)}$$

$$=(-1)^n \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{n+2s} \frac{1}{\lfloor s \; \Gamma(n+s+1)}$$

$$=(-1)^n \, J_n(x).$$

Work for the next day VIII.

1. Write down in fair hand the entire lecture of to-day.

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NINTH LECTURE.

[Recurrence formulae for $J_n(x)$]

1. n being a positive integer, we have

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

$$\therefore\quad x\,J_n'(x)=\sum_{r=0}^{\infty}\frac{(-1)^r\,(n+2r)}{r!\,(n+r)!}\cdot\left(\frac{x}{2}\right)^{n+2r}\quad\ldots\ldots(1)$$

$$=n\,J_n(x)+x\sum_{r=1}^{\infty}\frac{(-1)^r}{(r-1)!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r-1}$$

Putting $r-1=s$

$$=n\,J_n(x)-x\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,(n+s+1)!}\left(\frac{x}{2}\right)^{n+2s+1}$$

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$$=n\,J_n(x)-x\,J_{n+1}(x)$$

Again (1) can be written as

$$x\,J_n'(x)=-n\,J_n(x)+x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r-1)!}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=-n\,J_n(x)+x\,J_{n-1}(x).$$

Thus we have

$$x\,J_n'(x)=n\,J_n-x\,J_{n+1}\quad\ldots\ldots\ldots\ldots\ldots\ (I)$$

$$x\,J_n'(x)=-n\,J_n+x\,J_{n-1}\quad\ldots\ldots\ldots\ldots\ (II)$$

From the above we get by subtracting and adding

$$2n\,J_n=x\,(J_{n-1}+J_{n+1})\quad\ldots\ldots\ldots\ldots\ldots\ (III)$$

$$2J_n'(x)=J_{n-1}-J_{n+1}\quad\ldots\ldots\ldots\ldots\ldots\ (IV).$$

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hence we have

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{-\infty}^{\infty} t^n \; J_n(x) \ldots\ldots\ldots \quad (2)$$

21 Series for $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $(x \sin \phi)$, $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}$ $(x \cos \phi)$.

By putting $t=e^{i\phi}$ in (2) of Art 20

we have

$$e^{i\,x \sin \phi}=\sum_{n=-\infty}^{\infty} e^{i\,n\,\phi} \; J_n(x)$$

or $\cos(x \sin \phi)+i \sin(x \sin \phi)$

$$=J_0(x)+[J_1(x)\, e^{i\phi}+J_{-1}(x)\, e^{-i\phi}]+[J_2(x)\; e^{2i\phi}$$

$$+J_{-2}(x)\; e^{-2i\phi}\;]+[J_3(x)\; e^{3i\phi}$$

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$$+J_{-3}(x)\; e^{-3i\phi}\;]+\ldots\ldots\ldots\ldots$$

$$=J_0(x)+J_1(x)\;[e^{i\phi}-\;e^{-i\phi}]+J_2(x)\;[e^{2\,i\phi}+\;e^{-2i\phi}]$$

$$+J_3(x)\,[\;e^{3\;i\phi}-\;e^{-3\;i\phi}\;]+J_4(x)\;[\;\;e^{4\;i\phi}-\;e^{-4i\phi}]$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=J_0(x)+\;2\,i \sin \phi \; J_1(x)+2 \cos 2\,\phi \; J_2(x)$$

$$+2\,i \sin 3\,\phi \; J_3(x)+2 \cos 4\,\phi \;\; J_4(x)+\ldots\ldots\ldots$$

Equating real and immaginary parts we have

$$\cos(x \sin \phi)=J_0(x)+2 \cos 2\,\phi \; J_2(x)+2 \cos 4\;\phi \; J_4\;(x)$$
$$+\;\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

$$\sin(x \sin \phi)=2\; J_1(x) \sin \phi+2\; J_3(x) \sin 3\;\phi$$
$$+2\; J_5(x) \sin 5\;\phi+\ldots\;\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

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By replacing ϕ by $\frac{\pi}{2}-\phi$ in (1) and (2) we get

$$\cos(x\cos\phi)=J_0(x)-2\cos 2\phi\, J_2(x)+2\cos 4\phi\, J_4(x)$$
$$-2\cos 6\phi\, J_6(x)+\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$\sin(x\cos\phi)=2\cos\phi\, J_1(x)-2\cos 3\phi\, J_3(x)$$
$$+2\cos 5\phi\, J_5(x)-2\cos 7\phi\, J_7(x)+\ldots\ldots\ldots\ldots \quad (4)$$

Work for the next day X

1. Show that

$$\cos x=J_0(x)-2J_2(x)+2J_4(x)-\ldots\ldots\ldots$$

$$\sin x=2J_1(x)-2J_3(x)+2J_5(x)-\ldots\ldots\ldots$$ (Agra 1936)

[Hint put $\phi=0$ in (3) and (4) of Art 21].

2. $x\sin x=2[2^2 J_2(x)-4^2 J_4(x)+6^2 J_6(x)-\ldots\ldots\ldots\ldots]$

$x\cos x=2[1^2 J_1(x)-3^2 J_3(x)+5^2 J_5(x)-\ldots\ldots\ldots]$

[Hint Diff (3) and (4) of Art. 21 twice with respect to ϕ and then put $\phi=0$].

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ELEVENTH LECTURE

[Integrals for $J_0(x)$ and $J_n(x)$]

22. From (1) of Art 21 we get by integrating with respect to ϕ

$$\int_0^{\pi} \cos(x \sin \phi)\, d\phi = \pi J_0(x)$$

as other terms on the right hand side vanish between the limits of integration.

$$\therefore \quad J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin \phi)\, d\phi \ldots\ldots\ldots(A)$$

Again multiplying (1) of Art 21 by $\cos n\phi$ and integrating we get

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$$\int_0^{\pi} \cos(x \sin \phi) \cos n\phi\, d\phi = 0 \text{ or } \pi J_n(x)$$

according as n is odd or even.

Multiplying (2) of Art 21 by $\sin n\phi$ and integrating we have

$$\int_0^{\pi} \sin(x \sin \phi) \sin n\phi\, d\phi = 0 \text{ or } \pi J_n(x)$$

according as n is even or odd.

Hence on addition we get

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x \sin \phi)\, d\phi$$

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23. To prove

$$J_n(x)=\frac{2}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_0^{\pi}\cos(x\sin\phi)\cos^{2n}\phi\,d\phi.$$

Expanding cos (x sin ϕ) in powers of x sin ϕ the general term on the right hand side is

$$\frac{2}{\pi\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n(-1)^r\int_0^{\pi}\frac{x^{2r}}{\lfloor\underline{2r}}\sin^{2r}\phi\cos^{2n}\phi\,d\phi$$

Now $\displaystyle\int_0^{\pi}\sin^{2r}\phi\cos^{2n}\phi\,d\phi=2\int_0^{\pi/2}\sin^{2r}\phi\cos^{2n}\phi\,d\phi$

$$=\int_0^{1}t^{\frac{2r-1}{2}}(1-t)^{\frac{2n-1}{2}}dt$$ www.dbraulibrary.org.in

where $t=\sin^2\phi$.

$$=\frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(1+r+n)}$$

Hence the general term on the right hand side

$$=\frac{2}{\sqrt{\pi}}\cdot\frac{1}{\Gamma(n+\frac{1}{2})}\cdot\left(\frac{x}{2}\right)^n(-1)^r\cdot\frac{x^{2r}}{\lfloor\underline{2r}}\cdot$$

$$\frac{\frac{2r-1}{2}\cdot\frac{2r-3}{2}\ldots\frac{1}{2}\,\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})}{\Gamma(1+r+n)}$$

$$=(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{\lfloor\underline{r}}\,\frac{1}{\Gamma(n+r+1)}$$

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Thus we have

$$J_n(x) = \frac{2}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n \int_0^{\pi} \cos(x \sin\phi) \cos^{2n}\phi \, d\phi$$

Work for the next day XI.

1. Show that each of the following integrals is zero

$$\int_0^{\pi} \cos(x \sin\phi) \sin n\phi \, d\phi,$$

$$\int_0^{\pi} \sin(x \sin\phi) \cos n\phi \, d\phi, \qquad \int_0^{\pi} \sin(x \sin\phi \pm n\phi) \, d\phi$$

$$\int_0^{\pi} \cos(x \cos\phi) \sin n\phi \, d\phi, \qquad \int_0^{\pi} \sin(x \cos\phi) \sin n\phi \, d\phi$$

n being an integer

2. Show that

$$J_n(x) = \frac{(-1)^n}{\pi} \int_0^{\pi} \cos(x \cos\phi) \cos 2n\phi \, d\phi$$

(n being an integer)

$$J_n(x) = \frac{(-1)^n}{\pi} \int_0^{\pi} \sin(x \cos\phi) \cos(2n+1)\phi \, d\phi$$

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3. $\int_0^\pi \cos(x\cos\phi)\cos(2n+1)\phi\, d\phi=0.$

$$\int_0^\pi \sin(x\cos\phi)\cos 2n\phi\, d\phi=0$$

Note :— All the results of Questions 1, 2, 3 can be obtained from results of Art 21.

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TWELFTH LECTURE

[Integral properties of $J_n(x)$; zeroes of $J_n(x)$].

24 To show that $\int_0^a J_n(kr)\, J_n(k'r)\, dr = 0$

where k and k' are different roots of $J_n(xa)=0$

Special case of Green's theorem for a circle of radius a is

$$\int_0^a \int_0^{2\pi} (v\nabla^2 u - u\nabla^2 v)\, r\, d\theta\, dr$$

$$= a\int_0^{2\pi} \left(v\frac{\partial u}{\partial r} - u\frac{\partial v}{\partial r}\right) d\theta \quad r=a \quad \ldots\ldots\ldots\ldots (1)$$

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Also $\nabla^2 u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

in polar coordinates

Taking u to be $J_n(kr)\cos n\theta$ and v to be $J_n(k'r)\cos n\theta$ and knowing that J_n satisfies Bessel's equation we have

$\nabla^2 u = -k^2 u,\ \nabla^2 v = -k'^2 v$

Substituting in (1) we have

$$(k'^2 - k^2)\int_0^{2\pi}\int_0^a J_n(kr)\, J_n(k'r)\cos^2 n\theta\, r\, d\theta\, dr$$

$$= a\int_0^{2\pi} \cos^2 n\theta \left\{ k\, J_n(k'a)\frac{d}{da}J_n(ka) - k'\, J_n(ka)\frac{d}{da}J_n(k'a) = 0 \right\} d\theta = 0$$

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since $J_n(ka)=J_n(k'a)=0$.

Hence

$$(k'^2-k^2)\int_0^a J_n(kr)\,J_n(k'r)\,r\,dr=0$$

Hence the theorem since $k'\neq k$

25. All the roots of $J_n(x)=0$ are real.

If the theorem is not true then take two conjugate complex roots $\lambda+i\mu$ and $\lambda-i\mu$.

Put in the last result of the last article

$ka=\lambda+i\mu,\ k'a=\lambda-i\mu$

Then

$$4i\lambda\mu\int_0^a J_n(kr)\,J_n(k'r)\,r\,dr \ldots\ldots\ldots\ldots\ldots(1)$$

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But $J_n(kr)$ and $J_n(k'r)$ are conjugate complex quantities, say equal to $P+iQ$ and $P-iQ$ respectively.

Hence (1) becomes

$$4i\lambda\mu\int_0^a (P^2+Q^2)\,r\,dr=0$$

Since the integral is not zero as the integrand is throughout positive, we have $\lambda\mu=0$, but λ cannot be zero as then a purely imaginary quantity $i\mu$ will have to satisfy $J_n(x)=0$ which is impossible since the sum of decidedly positive quantities cannot be zero, it therefore follows that $\mu=0$.

Hence the theorem.

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26. $J_n(x)=0$ has no repeated roots except $x=0$

We have from (I) of Art. 19

$$J'_n = \frac{n}{x}J_n - J_{n+1} \quad \ldots\ldots\ldots\ldots(1)$$

Suppose α is a repeated root of J_n

then $J_n(\alpha)=0$, $J'_n(\alpha)=0$

so that from (1) $J_{n+1}(\alpha)=0$

From III of Art. 19, $J_{n-1}(\alpha)=0$

Thus for the same value α of x, $J_n(x)$, $J_{n+1}(x)$, $J_{n-1}(x)$ are all equal to zero, which is absurd as we can not have two power series having the same sum function. Thus there can not be any repeated root of J_n except $x=0$.

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Work for the Next day XII.

1. Prove that

$$J_3(x)+3J_0'(x)+4J_0'''(x)=0$$

2. Prove that

$$\frac{x}{2}J_n=(n+1)J_{n+1}-(n+3)J_{n+3}+(n+5)J_{n+5}-\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Work for the next day XIII.

1. Prove that

$$J_1(x)+J_3(x)+J_5(x)+\ldots\ldots\ldots\ldots$$

$$=\tfrac{1}{2}\left[J_0(x)+\int_0^x\left\{J_0(t)+J_1(t)\right\}dt-1\right]$$

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2. Prove that

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \dots \dots \dots \dots$$

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \dots \dots \dots$$

[Hint put $\phi=0$ in (III) and (IV) of Art 21).

3. Show that

$$x \sin x = 2[2^2 J_2(x) - 4^2 J_4(x) + 6^2 J_6(x) - \dots \dots \dots]$$

$$x \cos x = 2[1^2 J_1(x) - 3^2 J_2(x) + 5^2 J_5(x) - \dots \dots \dots$$

[Hint differentiate (3) and (4) of Art. 21 twice with respect to x and then put $\phi=0$]

4. Show that

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + \dots \dots \dots \dots$$

$$x = 2J_1(x) + 2 \,.\, 3J_3(x) + 2 \,.\, 5J_5(x) + \dots \dots \dots \dots$$

[Hint for the first result put $\phi=0$ in (1) of Art. 21 for he second result divide (2) of Art 21 by ϕ and then put $\phi=0$].

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PARTIAL DIFFERENTIAL

EQUATIONS OF SECOND

AND

Higher Orders.

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THIRTEENTH LECTURE

[Linear Partial Differential Equations with constant coefficients

27. Generally x and y will be the independent variables d z the dependent variable.

$\frac{\partial^n z}{\partial x^m \partial y^{n-m}}$ $(m=0, 1, 2, \ldots\ldots\ldots, n)$ are partial differential oefficients of z of order n.

A partial differential equation in which the dependent variable and its partial differential coefficients of various orders occur linearly and in which the coefficients are not functions of the dependent variables but merely constants s called *linear partial differential equation with constant coefficients*. Such an equation is

$$\phi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) z = V$$

where ϕ is a rational integral algebraic function, all the coefficients of which are constants and V is any function of the dependent variables

As in the case of ordinary linear differential equations with constant coefficients the complete integral consists of the sum of two parts, the first of which is the most general integral of

$$\phi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) z = 0$$

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and the second any particular solution of

$$\phi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) z = V.$$

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are denoted by D and D' respectively.

28. Let us take the case of the equation in which all the differential coefficients that occur are of the nth order so that it is

$$(D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \ldots\ldots\ldots + A_n D'^n) z = V$$

where V is a function of x and y

or $(D - \alpha_1 D')(D - \alpha_2 D') \ldots\ldots\ldots (D - \alpha_n D') z = V$

where $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ are the roots of the equation

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$$\xi^n + A_1 \xi^{n-1} + A_2 \xi^{n-2} + \ldots\ldots\ldots + A \xi^n = 0$$

The complementary function is the most general solution of the equation

$$(D - \alpha_1 D')(D - \alpha_2 D') \ldots\ldots\ldots (D - \alpha_n D') z = 0$$

The n solutions of the above equation are the solutions of the n differential equations

$$(D - \alpha_r D') z = 0 \; [r = 1, 2, \ldots\ldots\ldots, n].$$

$z = C_r e^{\alpha_r D'} x$ is the solution of $(D - \alpha_r D') z = 0$

$\alpha_r D'$ being considered as a more algebraic quantity and C_r being a quantity independent of x. In its most general form C_r is an arbitrary function of y as y is independent of x.

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Thus the solution is

$$z=\phi_r(y)\, e^{\alpha_r D' x} = \phi_r(y).\, e^{\alpha_r x. D'}$$

$$=\phi_r(y)\left[1+\alpha_r x D' + \frac{(\alpha_r x)^2}{\lfloor 2} D'^2 + \dots\dots\right]$$

$$=\phi_r(y) + \alpha_r x \phi_r'(y) + \frac{\alpha_r^2 x^2}{\lfloor 2}\phi_r''(y) + \dots\dots + \dots\dots$$

$$=\phi_r(y+\alpha_r x).$$

Hence the complementary function is

$$z=\sum_{r=1}^{r=n} \phi_r(y+\alpha_r x).$$

where ϕ_r is arbitrary.

If any root say α_1 is a double root, then the part of the complementary function corresponding to this root is the solution of the equation

$$(D-\alpha_1 D')^2 z=0$$

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Which is

$$z=(A+Bx)\, e^{(\alpha_1 D')x} = (A+Bx)\, e^{\alpha_1 x . D'}$$

[D' being considered as an algebraic quantity and A and B being arbitrary function of y say $\phi_1(y)$ and $\phi_2(y)$].

$$\text{or } z=\{\phi_1(y)+x\,\phi_2(y)\}\left[1+\alpha_1 x D' + \frac{(\alpha_1 x)^2}{\lfloor 2} D' + \dots\dots\right]$$

$$\text{or } z=\phi_1(y+\alpha_1 x)+x\,\phi_2(y+\alpha_1 x)$$

If α_1 is repeated three times; the part of the complementary function corresponding to it would be

$$\phi_1(y+\alpha_1 x)+x\,\phi_2(y+\alpha_1 x)+x^2\,\phi_3(y+\alpha_1 x)$$

And so on exactly on the pattern of ordinary linear differential equations with constant coefficients.

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Work for the next day XIII

Solve the equations

1. $$\frac{\partial^2 z}{\partial x^2} - 3\,\frac{\partial^2 z}{\partial x\,\partial y} + 2\,\frac{\partial^2 z}{\partial y^2} = 0$$

2. $$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

3. $$\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$$

4. $$\frac{\partial^4 z}{\partial x^4} - 2\,\frac{\partial^4 z}{\partial x^3\,\partial y} + 2\,\frac{\partial^4 z}{\partial x\,\partial y^3} - \frac{\partial^4 z}{\partial y^4} = 0.$$

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FOURTEENTH LECTURE

28. To find the particular integral of

$(A_0 D^n + A_1 D^{n-1}D' + A_2 D^{n-2}D'^2 + \dots\dots\dots + A_n D'^n) z = V$

or of $(D-\alpha_1 D')(D-\alpha_2 D')\dots\dots\dots(D-\alpha_n D') z = V$

We have

$$z = \frac{V}{(D-\alpha_1 D')(D-\alpha_2 D')(D-\alpha_3 D')\dots\dots(D-\alpha_n D')}$$

$$= \frac{V}{D'^n \left(\frac{D}{D'}-\alpha_1\right)\left(\frac{D}{D'}-\alpha_2\right)\dots\dots\left(\frac{D}{D'}-\alpha_n\right)}$$

$$= \frac{1}{D'^n}\sum_{r=1}^{r=n} \frac{N_r \,.\, V}{\left(\frac{D}{D'}-\alpha_r\right)}$$ where N_r is a constant.

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$$= \sum_{r=1}^{r=n} \frac{1}{D'^{n-1}} \cdot \frac{N_r\, \psi(x, y)}{(D-\alpha_r D')}$$

where $V = \psi(x, y)$ ... ... (A)

Now $\frac{\psi(x, y)}{(D-\alpha_r D')} = e^{\alpha_r x D'} \int \psi(x, y)\, e^{-\alpha_r x D'}\, dx$

$$= e^{\alpha_r x D'} \int^x \psi(\xi, y)\, e^{-\alpha_r \xi D'}\, d\xi$$

$$= e^{\alpha_r x D'} \int^x \psi(\xi, y-\alpha_r \xi)\, d\xi$$

$$= \int^x \psi(\xi, y-\alpha_r \xi + \alpha_r x)\, d\xi$$

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Hence

$$z=\int\int\int\ldots\ldots dy^{n-1}\left[\sum_{r=1}^{n} N_r \, . \int^{x} \psi\,(\xi\; y+\alpha_r\; x-\alpha_r\; \xi)\, d\xi\right.$$

This is the most general form of the solution. In particular cases the working becomes much more easy. Thus if V is a function of x only we may consider $[\phi\,(D,\,D')]^{-1}$ expanded in a series of ascending powers of D' and only such terms need be considered as do not contain D'. Corresponding simplifications arise in other special cases exactly on the pattern of ordinary linear differential equations with constant coefficients.

Example 1. Solve :

$$\frac{d^2u}{dx^2}-m^2\,\frac{d^2u}{dy^2}=x^2$$

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The complementary function is

$$z=\phi\,(y-mx)+\psi\,(y+mx)$$

where ϕ and ψ are arbitrary functions.

For the particular integral we have

$$u=\frac{x^2}{D^2-m^2D'^2}=\frac{x^2}{D^2}\left(1-\frac{m^2D'^2}{D^2}\right)^{-1}$$

$$=\frac{x^2}{D^2}\left(1+\frac{m^2D'^2}{D^2}+\ldots\ldots\right)$$

$$=\frac{x^2}{D^2}=\frac{x^4}{3.4.}$$

hence the complete integral is

$$u=\phi\,(y+mx)+\psi\,(y-mx)+\frac{x^4}{3.4.}$$

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Work for the next day XIV

1. Obtain a solution of the quation

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$$

ch that, when $t=0$, $y=F(x)$ and $\frac{\partial y}{\partial t}=\frac{d f(x)}{dx}$, $F(x)$ and $x)$ being known functions of x.

2. $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\cos mx \cos ny.$

3 $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\sin x,$

4. $\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}=\sin x \cos y.$

5. $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x\,\partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$

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Work for the next day XV

1. $\frac{\partial^2 z}{\partial x^2}-2a\frac{\partial^2 z}{\partial x\,\partial y}+a^2\frac{\partial^2 z}{\partial y^2}=f(y+ax)$

2. $\frac{\partial^3 z}{\partial x^3}-\frac{\partial^3 z}{\partial y^3}=x^3y^3$

3. $(\phi-aD')^3 z=\phi(x)+\psi(y)+K(x+by).$

4. $(D-D')^2 z=x+\phi(x+y).$

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FIFTEENTH LECTURE

29. Let us solve the equation

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 u}{\partial z^3}-3\ \frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=x^3+y^3+z^3-3x\ z.$$

Where the meanings of D's are obvious. For the complementary function we have

$$(D_1^3+D_2^3+D_3^3-3D_1\,D_2\,D_3)\,u=x^3+y^3+z^3-3xyz$$

or $(D_1+D_2+D_3)(D_1+\omega\,D_2+\omega^2\,D_3)(D_1+\omega^2\,D_2+\omega D_3)u=0$

ω being a cube root of unity.

Now $(D_1+\alpha D_2+\beta D_3)u=0$

gives $u=Ae^{(-\alpha\,D_2-\beta D_3)x}$

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where A is $\phi\,(y, z)$, ϕ being arbitrary

or $u=\phi\,(y, z)\,e^{(\alpha\,D_2-\beta D_3)x}$

$=\phi\,(y-\alpha x,\ z-\beta x)$

Hence the complementary function is

$$\phi_1(y-x,\ z-x)+\phi_2\,(y-\omega x,\ z-\omega^2 x)+\phi_3\,(y-\omega^2 x,\ z-\omega x)$$

The part of the particular integral corresponding to x^3 is

$$\frac{x^3}{D_1^3+D_2^3+D_3^3-3D_1\,D_2\,D_3}=\frac{x^3}{D_1^3}=\frac{x^6}{4.5.6}.$$

and so for y^3 and z^3

The part corresponding $-3xyz$ is

$$\frac{-3xyz}{-3\,D_1\,D_2\,D_3}=\frac{x^2y^2z^2}{8}$$

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The complete integral is the sum of the complementary function and the particular integral.

30. Let us consider the solution of the equations

(i) $\frac{\partial z}{\partial x} - \alpha \frac{\partial z}{\partial y} - \beta z = 0$

or $(D - \alpha D' - \beta) z = 0$

then $z = e^{(\alpha D' + \beta) x} . \psi(y)$

D′ being considered algebraic and ψ being arbitrary

$= e^{\beta x} \psi(y) . e^{\alpha x D'}$

$= e^{\beta x} \psi(y + \alpha x).$

(ii) $(D - \alpha D' - \beta)^r z = 0,$

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Then $z = e^{(\alpha D' + \beta) x} [A_1 + A_2 x + A_3 x^2 + \ldots\ldots + A_n x^{n-1}]$

where A_1, A_2, A_3, ... A_n, are arbitrary functions of y.

$= e^{\beta x} [\phi_1(y + \alpha x) + x \phi_2(y + \alpha x) + \ldots\ldots\ldots\ldots\ldots + x^{n-1} \phi_n(y + \alpha x)].$

(iii) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3 \frac{\partial z}{\partial x} + 3 \frac{\partial z}{\partial y} = xy + e^{x+2y}$

For the complementary function we have

$(D - D')(D + D' - 3) z = 0$

Part of the complementary function corresponding to the first factor is $\phi_1(y + x)$

and the part corresponding to the second factor is

$e^{3x} \phi_2(y - x).$

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Hence the complementary function is

$$\phi_1(y+x)+e^{3x}\phi_2(y-x).$$

where ϕ_1 and ϕ_2 are arbitrary.

The part of the particular integral corresponding to e^{x+2y} is

$$z=\frac{1}{(D-D')(D+D'-3)}\cdot e^{x+2y}$$

$$=e^{x}\cdot\frac{1}{(1-D')(D'-2)}e^{2y}$$

$$=e^{x+2y}\frac{1}{(1-D'-2)(D'+2-2)}\cdot 1$$

$$=-e^{x+2y}\cdot\frac{1}{D'(D'+1)}\cdot 1.$$

$$=-ye^{x+2y}$$

And the part corresponding to the term xy is

$$\frac{xy}{(D-D')(D+D'-3)}$$

$$=-\frac{1}{3}\frac{1}{D-D'}\left\{1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\cdots\cdots\right\}xy$$

$$=-\frac{1}{3D}\left(1+\frac{D'}{D}+\cdots\right)\left\{xy+\frac{x+y}{3}+\frac{2}{9}\right\}$$

$$=-\frac{1}{3D}\left(xy+\frac{x+y}{3}+\frac{2}{9}+\frac{x^2}{2}+\frac{x}{3}\right)$$

$$=-\frac{1}{3}\left[\frac{yx^2}{2}+\frac{x^2}{6}+\frac{xy}{3}+\frac{2}{9}x+\frac{x^3}{6}+\frac{x^2}{6}\right]$$

$$=-\left(\tfrac{1}{6}x^2y+\tfrac{1}{9}x^2+\tfrac{1}{9}xy+\tfrac{2}{27}x+\tfrac{1}{18}x^3\right)$$

The complete integral is the sum of the complementary function and the particular integral.

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It will be noticed that by a different method of procedure such as expanding in powers of D/D′ we get a particular integral different in form from the one we have obtained above but the two could be transformed into each other by the help of the complementary function.

30. Equations in which the coefficient of a differential equation of any order is a constant multiple of the variables of the same degree, may be reduced to an equation of the type we have already discussed. Such an equation is of the form

$$\Sigma A_r x^r \frac{\partial^r z}{\partial x^r} + \Sigma B_{p,q} \frac{\partial^{p+q} z}{\partial x^p \partial y^q}$$

$$+ \Sigma C_s Y^s \frac{\partial^s z}{\partial y^s} = 0$$

We may change the independent variables to u and v where $x = e^u$, $y = e^v$ or which is the same thing we write

$$x^r \frac{\partial^r z}{\partial x^r} = \odot(\odot-1)(\odot-2)\ldots\ldots(\odot-r+1)z$$

$$x^p y^q \frac{\partial^{p+q} z}{\partial x^p \partial y^q} = \odot(\odot-1)\ldots\ldots(\odot-p+1)\phi(\phi-1)\ldots\ldots\ldots$$

$$(\phi-q+1)\, z$$

$$y^s \frac{\partial^s z}{\partial y^s} = \phi(\phi-1)(\phi-2)(\phi-3)\ldots\ldots(\phi-s+1)\, z$$

Where $\odot$ represents $x\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial u}$ and ϕ

represents $y\frac{\partial}{\partial y}$ or $\frac{\partial}{\partial v}$

If we denote $\frac{\partial}{\partial u}$ by D and $\frac{\partial}{\partial v}$ by D'

we get the equation of the type already discussed.

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Example :—Solve the equation

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = x^m y^n$$

putting $x = e^u, \; y = e^v$

so that $x\frac{\partial}{\partial x} = \frac{\partial}{\partial u} = \theta; \; y\frac{\partial}{\partial y} = \frac{\partial}{\partial v} = \phi$

hence

$$\{\theta(\theta-1) + 2\theta\phi + \phi(\phi-1)\} = 0$$

for complementary functions.

or $\quad (\theta+\phi)(\theta+\phi-1)\, z = 0$

so that $\quad z = F_1(v-u) + e^u F_2(v-u)$

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$$= \psi\left(\frac{y}{x}\right) + xK\left(\frac{y}{x}\right)$$

where ψ and K are arbitrary.

$$z = \frac{e^{mu+nv}}{(\theta+\phi)(\theta+\phi-1)} = \frac{e^{mu+nv}}{(m+n)(m+n-1)}$$

$$= \frac{x^m y^n}{(m+n)(m+n-1)}$$

The complete integral is therefore

$$z = \psi\left(\frac{y}{x}\right) + xK\left(\frac{y}{x}\right) + \frac{x^m y^n}{(m+n)(m+n-1)}$$

—:0:—

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Work for two days XV.

1. Solve

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial y\partial z}=xyz$$

2. $$\frac{\partial^3 u}{\partial x^2\partial y}-2\frac{\partial^3 u}{\partial x\partial y^2}-3\frac{\partial^3 u}{\partial x^2\partial z}-3\frac{\partial^3 u}{\partial x\partial z^2}-2\frac{\partial^3 u}{\partial y^2\partial z}$$
$$+6\frac{\partial^3 u}{\partial y\partial z^2}+7\frac{\partial^3 u}{\partial x\partial y\partial z}=0$$

3. $$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=xy$$

4. $$x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=y\frac{\partial z}{\partial y}-x\frac{\partial z}{\partial x}$$

5. $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+nu$$
$$=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)+x^2+y^2+x^2$$

6. $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(x^2+y^2)^{\frac{1}{2}n}$$

7. $$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^2 u+n^2u=0.$$

8. $$\frac{\partial^2 z}{\partial x^2}-a^2\frac{\partial^2 z}{\partial y^2}+2ab\frac{\partial z}{\partial x}+2a^2b\frac{\partial z}{\partial y}=0$$

9. $$mn\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)-(m^2+n^2)\frac{\partial^2 z}{\partial x\partial y}+mn\left(n\frac{\partial z}{\partial x}-m\frac{\partial z}{\partial y}\right)$$
$$=\cos(mx+ny)+\cos(kx+ly)$$

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SIXTEENTH LECTURE

[Partial Differential Equations of the second ord Monge's Method].

31. As mentioned before z is to be regarded as t dependent variable and x and y as the independent va ables. Partial differential coefficients $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are denot by p and q while $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \, \partial y}$, $\frac{\partial^2 z}{\partial y^2}$ are denoted by r, s and respectively. It may be noted that

$$\frac{\partial p}{\partial x}=r, \; \frac{\partial q}{\partial y}=t, \; \frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}=s.$$

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Def. An equation is said to be a *partial differenti equation of the second Order* when it includes one at least the partial differential coefficients r, s and t but none of higher order. The quantities x, y, p and q generally occ in the equation but it is not necessary

The most general form of partial differential coefficie of the second order is

$$F(x, y, z, p, q, r, s, t)=)$$

Def. The *Complete integral* of the equations is a relati between x, y, z such that when the value of z derived fro it and the partial differential coefficients formed from z a substituted in the differential equation, the latter becom an identity. The complete integral may be of differe forms. It may involve arbitrary constants or arbitra functions or both.

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Def. An intermediate integral of an equation of the second order is a partial differential equation of the first order such that the given differential equation can be derived from it. Integration of this equation gives the complete integral of the original equation.

The intermediate integral can involve only one arbitrary function. Its integration will add one more so that the complete integral will involve two arbitrary functions.

32. Monge's method of solution of the equation

$$Rr+Ss+Tt=V \quad \ldots\ldots\ldots\ldots(1)$$

where r, s, t have their usual meanings and R, S, T and V are functions of x, y, z, p and q.

Assuming that the equation is integrable we substitute the values of r and t as found from

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$$dp=r\,dx+s\,dy \left(dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy\right)$$

$$dq=s\,dx+t\,dy \left(dq=\frac{\partial q}{\partial x}dx+\frac{\partial q}{\partial y}dy\right)$$

in equation (1) and it transforms into

$$R\,dp\,dy+T\,dq\,dx-V\,dx\,dy=s(R\,dy^2-S\,dx\,dy+T\,dx^2) \ldots\ldots(2)$$

The subsidiary equations are

$$R\,dy^2-S\,dx\,dy+T\,dx^2=0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$R\,dp\,dq+T\,dq\,dx-V\,dx\,dy=0 \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

(3) resolves into two linear equations in dx and dy such as

$$dy-\xi_1\,dx=0,\ dy-\xi_2\,dx=0.$$

One of these equations combined with (4) and with $dz=p\,dx+q\,dy$ if necessary may lead to two integrals $u=\alpha$

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and $v=b$ which may give.

$u_1=f(v_1)$ as the intermediate integral of (1).

Proceeding similarly with the other of the two linear equations we may get

$u_2=f(v_2)$ as the *other* intermediate integral.

From the two intermediate integrals we find the values of p and q in terms of x and y and then substituting these values in $dz=pdx+qdy$

and integrating it we get the complete integral of the original equation.

In case only one intermediate integral say $u_1=f(v_1)$ is deducible and not the other, we proceed to integrate it by methods which apply to partial differential equations of the first order and thus get the final integral.

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Example 1. Solve $r=a^2t$..........................(1)

We know

$$dp=rdx+sdy,\ dq=sdx+tdy.$$

so that

$$r=\frac{dp-sdy}{dx},\ t=\frac{dq-sdx}{dy}$$

Substituting these values we get

$$\frac{dp-sdy}{dx}=a^2\frac{dq-sdx}{dy}$$

or $dp\,dy-a^2dx\,dq=s(dy^2-a^2dx^2)$

The subsidiary equations therefore are

$dy^2-a^2dx^2=0,\ dp\,dy-a^2dx\,dq=0$..................(2)

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The first of these equations resolves into linear equations

$$\left.\begin{array}{l} dy-adx=0 \\ dy+adx=0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Taking the first of these and combining it with the second of the subsidiary equations we get

$$dp-adq=0.$$

Thus we have *simultaneously*

$$dy-adx=0$$

$$dp-adq=0.$$

or, $y-ax=A,$

$$p-aq=B.$$

Therefore $p-aq=\phi_1(y-ax)$ ....(α)

is an intermediate integral.

Taking the second of (3) and combining it with the second of the subsidiary equations we get

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$$y+ax=A'$$

$$p+aq=B'$$

which give

$$p+aq=\phi_2(y+ax) \ldots\ldots\ldots\ldots\ldots\ldots(\beta)$$

as the second intermediate integral.

Substituting the values p and q as found from (α) and (β) in the relation $dz=pdx+qdy,$

we get

$$dz=\tfrac{1}{2}\left[\,\phi_1(y-ax)+\phi_2(y+ax)\,\right]dx$$

$$+\frac{1}{2a}\Big[\phi_2(y+ax)-\phi_1(y-ax)\Big]dy$$

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$$\text{or,}\quad dz=\frac{dy+adx}{2a}\,\phi_2(y+ax)-\frac{dy-adx}{2a}\,\phi_1(y-ax)$$

which on integration gives

$$z=\phi\,(y+ax)+\psi\,(y-ax)$$

where

$$\phi\,(t)=\frac{1}{2a}\int\phi_2\,(t)\,dt,\quad \psi\,(t)=-\frac{1}{2a}\int\phi_1\,(t)\,dt$$

ϕ and ψ are arbitrary since ϕ_1 and ϕ_2 are abitrary and the arbitrary constant of integration may be supposed to be absorbed in either of the functions ϕ and ψ.

Example 2. Solve :—

$$(b+cq)^2r-2(b+cq)(a+cp)s+(a+cp)^2t=0$$

Putting $r=\dfrac{dp-sdy}{dx}$, $t=\dfrac{dq-sdx}{dy}$

the subsidiary equations are

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$$(b+cq)^2\,dy^2+2(b+cq)(a+cp)\,dx\,dy+(a+cp)^2\,dx^2=0\;\ldots\ldots(1)$$

$$(b+cq)^2\,dp\,dy+(a+cp)^2\,dq\,dx=0\;\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

(1) Can be written as

$$(b+cq)\,dy+(a+cp)dx=0 \qquad (1)$$

Combining it with (2) we have

$$(b+cq)dp-(a+cp)dq=0$$

$$\text{or}\qquad \frac{dp}{a+cp}=\frac{dq}{b+cq}$$

or $(a+cp)=A\,(b+cq)$ by integration. (3)

Combining (1) with $dz=pdx+qdy$

we have

$$ax+by+cz=B. \qquad (4)$$

$\therefore$ from (3) and (4)

$$a+cp=(b+cq)\,\phi\,(ax+by+cz) \qquad (5)$$

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Denoting $\phi\ (ax+by+cz)$ by ϕ and applying Lagrange's rule we have

$$\frac{dx}{c}=\frac{dy}{-c\phi}=\frac{dz}{-a+b\phi}$$

$$=\frac{a\,dx+b\,dy+c\,dz}{0}$$

$\therefore\ adx+bdy+cdz=0,\ ax+by+cz=C$

Hence from the first of these equations.

$y+\phi\ (C)\ x=C'=\psi\ (C)$ say

Hence

$y+x\phi\ (ax+by+c)=\psi\ (ax+by+c)$

is the final integral of our differential equation.

Work for two days XVI

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Solve the equations by Monge's method

1 $x^2r+2xys+y^2t=0$

2 $q^2r-2pqs+p^2t=0$

3 $x^2r-y^2t=0$

4 $r-a^2t+2ab\ (p+aq)=0$

5 $r-2as+a^2t=0$

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SEVENTEENTH LECTURE

33. Integration of the differential equation

$$Rr+Ss+Tt+U(rt-s^2)=V$$

where the capital letters denote function of

$$x, \quad y, \quad z, \quad p, \quad q.$$

Form the equation in λ

$$\lambda^2 (RT+UV)+\lambda US+U^2=0 \quad \text{.............(1)}$$

Let λ_1 and λ_2 be the two values of λ satifying thi equation

We may obtain two integrals $u_1=a_1$, $v_1=b_1$ of th equations

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$$\left.\begin{aligned} Udy+\lambda_1 Tdx+\lambda_1 Udp=0 \\ Udx+\lambda_2 Rdy+\lambda_2 Udq=0 \end{aligned}\right\} \text{.............(2)}$$

or we may obtain two integrals

$u_2=a_2$, $v_2=b_2$ of the equation

$$\left.\begin{aligned} Udy+\lambda_2 Tdx+\lambda_2 Udp=0 \\ Udx+\lambda_1 Rdy+\lambda_1 Udq=0 \end{aligned}\right\} \text{.............(3)}$$

We thus obtain two intermediate integrals $u_1=f_1(v_1)$ ar $u_2=f_2(v_2)$ and substituting in $dz=pdx+qdy$ the values of and q found from the two intermediate integrals and inte rating we get the complete integral of our differential equati

34. It may not be possible to obtain from the tv intermediate integrals the values of p and q suitable f insertion in

$$dz=pdx+qdy$$

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and in that case we may take one of the intermediate integrals $u_1 = f_1(v_1)$ and $u_2 = a$,

where $u_2 = a$ one of the integrals involved in the second intermediate integral.

The values of p and q derived from these will render the equation $dz = pdx + qdy$ integrable and thus we shall be able to get the final integral.

35. In case the two roots of the equation in λ are equal we get only one intermediate integral

$$u_1 = f(v_1)$$

which together with one of the first integrals say $v_1 = b$ will give the values of p and q suitable for insertion in

$$dz = pdx + qdy$$

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which after integration gives the final integral.

Example 1. Integrate the equation

$$ar + bs + ct + e(rt - s^2) = h$$

Here $R = a,\ S = b,\ T = c,\ U = e,\ V = h.$

The equation in λ is

$$\lambda^2 (RT + UV) + \lambda (US) + U^2 = 0$$

or $\lambda^2 (ac + eh) + \lambda . be + e^2 = 0.$ .... (1)

Putting $\lambda = -\frac{e}{m}$ ... ... (3)

(1) becomes.

$$\frac{e^2}{m^2}(ac + eh) - \frac{e^2 b}{m} + e^2 = 0$$

or $m^2 - bm + (ac + eh) = 0$ ... ... (3)

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an equation which determines m and hence by (2), λ.

Let the roots of (3) be m_1 and m_2

The first system of integrals is given by

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0$$

$$Udx+\lambda_2 Rdy+\lambda_2 Udq=0$$

or

$$edy+\left(-\frac{e}{m_1}\right)cdx+\left(-\frac{e}{m_1}\right)edp=0$$

$$edx+\left(-\frac{e}{m_2}\right)ady+\left(-\frac{e}{m_2}\right)edq=0$$

or

$$cdx+edp-m_1\ dy=0$$

$$ady+edq-m_2dx=0$$

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So that one intermediate integral is

$$cx+ep-m_1y=F\ (ay+eq-m_2x)\ \dots\ \dots\ \dots\ (4)$$

The second system of integrals is given by equatio[ns] which differ from the above equations in having m_2 in pla[ce] of m_1 and m_1 in place of m_2

Hence the second intermediate integral is

$$(cx+ep-m_2y)=\phi\ (ay+eq-m_1x)$$

As it is not possible to determine in practice the values o[f p] and q in terms of x and y from the two intermediate integra[ls] we combine any particular integral of the second system s[ay] $cx+ep-m_2y=A$ with (4) which is the general integral of t[he] first system.

Thus we have

$$(m_2-m_1)\ y+A=F\ (ay+eq-m_2x)$$

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or

$$ay+eq=m_2x+\psi\{(m_2-m_1)y+A\}$$

where ψ is inverse function of F and therefore as much arbitrary.

From this q is determinable and also p from

$$cx+ep-m_2y=A.$$

Substituting these values in $pdx+qdy=dz$, we have

$$edz=(A-cx+m_2y)dx$$
$$+[-ay+m_2x+\psi\{(m_2-m_1)y+A\}]dy$$

the integral of which is

$$ez+\frac{cx^2}{2}+\frac{ay^2}{2}+m_2xy+Ax+G\{(m_2-m_1)y+A\}+B.$$

where $G(t)=\frac{\int\psi(t)\,dt}{(m_2-m_1)}$ www.dbraulibrary.org.in

G being arbitrary since ψ is arbitrary.

Thus the complete integral of our differential equation has been obtained.

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Work for the next day XVIII

1. Solve $s^2-rt=a^2$
2. Solve $qr+(p+x)s+yt=-q+y(s^2-rt)$

EIGHTEENTH LECTURE

35. Example. Solve :—

$$z(1+q^2)r-2pqzs+z(1+p^2)t-z^2(s^2-rt)+1+p^2+q^2=0$$

Here

$$R=z(1+q^2),\ S=-2pqz,\ T=z(1+p^2)$$
$$U=z^2,\ V=-(1+p^2+q^2)$$

The equation in λ is

$$(RT+UV)\lambda^2+\lambda US+U^2=0$$

Or

$$z^2p^2q^2\lambda^2-2z^3pq\lambda+z^4=0$$

Or

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$$p^2q^2\lambda^2-2zpq\lambda+z^2=0$$

so that the two values of λ are identical each being $\dfrac{z}{pq}$

The two system of integrals reduce to one only namely that given by

$$Udy+\lambda Tdx+\lambda Udp=0$$
$$Udx+\lambda Rdy+\lambda Udq=0$$

Or by

$$z^2dy+\frac{z}{pq}\cdot z(1+p^2)\,dx+\frac{z}{pq}\,z^2\,dp=0$$

$$z^2dx+\frac{z}{pq}\,z(1+q^2)\,dy+\frac{z}{pq}\,z^2\,dq=0$$

Or by

$$pq\,dy+(1+p^2)dx+zdp=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

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$$pq\,dx+(1+q^2)dy+z\,dq=0 \qquad \ldots (2)$$

From (1) we have with the help of

$$dz=p\,dx+q\,dy$$

$$dx+p\,dz+z\,dp=0$$

or $\quad x+pz=A \qquad \ldots (3)$

From (2) we have similarly

$$y+qz=B \qquad \ldots (4)$$

Thus $F(x+pz,\ y+qz)=0$ is the intermediate integral.

We have

$$z\,dz=pz\,dx+qz\,dy$$

$$=(A-x)dx+(B-y)dy \text{ from (3) and (4)}$$

hence

$$(x-A)^2+(y-B)^2+z^2=C^2$$

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is the complete integral of our differential equation, A, B C being arbitrary constants.

Work for the next day XVIII

Solve :—

1. $xqr+ypt+xy(s^2-rt)=pq$

2. $(1+q^2)\,r-2pqs+(1+p^2)\,t$

$$=(s^2-rt)(1+p^2+q^2)^{-\frac{1}{2}}-(1+p^2+q^2)^{\frac{3}{2}}$$

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AGRA UNIVERSITY EXAMINATION PAPERS

1952

1. Define P_n and show that

(i) $$\frac{dP_n}{dx} - \frac{dP_{n-2}}{dx} = (2n-1)\, P_{n-1}$$

(ii) $$\int_{-1}^{1} P_m(x)\, P_n(x) = 0$$

where m and n are different positive integers and that

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

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2. Define $J_n(x)$ and show that

$$J_{n-1} + J_{n+1} = \frac{2}{x} n\, J_n$$

Prove that when n is a positive integer $J_n(x)$ is the co-efficient of z^n in the expansion of $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ in ascending and descending powers of z and can be expressed as

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta)\, d\theta$$

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3. Define a linear partial diffierential equation and explain Lagrange's Method of solving it

(i) Solve $(t+y+z)\frac{\partial t}{\partial x}+(t+x+z)\frac{\partial t}{\partial y}+(t+x+y)\frac{\partial t}{\partial z}=x+y+z$

(ii) Solve $\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x \partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$

4. Explain Monge's method of integration of the equation

$$Rr+Ss+Tt=V$$

Apply this method to integrate the equation

$$ar+bs+ct+e(rt-s^2)=h$$

Where a, b, c, e and h are constants.

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1953

1. (a) Define P_n and show that

$$nP_n=(2n-1)xP_{n-1}-(n-1)P_{n-2}$$

(b) Prove that

$$P_n=\frac{1}{\pi}\int_0^\pi \{x\pm(x^2-1)^{\frac{1}{2}}\cos\theta\}^n d\theta,$$

n being a positive integer throughout.

2. (a) Solve the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$$

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